Some supplements for tracking proofs.

Bhattacharyya coefficient between the target model $\hat{q}$ and the target candidate $\hat{p}$

$$\rho(\hat{p}, \hat{q}) = \sum_{u=1}^{m} \sqrt{\hat{p}_u(y)\hat{q}_u}$$

is the cosine of the angle between the $m$-dimensional unit vectors

$$\hat{p}(y) = [\sqrt{\hat{p}_1(y)}, \ldots, \sqrt{\hat{p}_m(y)}]^T \quad \hat{q} = [\sqrt{\hat{q}_1}, \ldots, \sqrt{\hat{q}_m}]^T$$

$$\cos(\hat{p}(y), \hat{q}) = \frac{\sum_{i=u}^{m} \sqrt{\hat{p}_u(y)\hat{q}_u}}{\sqrt{\sum_{u=1}^{m}[\sqrt{\hat{p}_u(y)}]^2} \sqrt{\sum_{u=1}^{m}[\sqrt{\hat{q}_u}]^2}}$$

and we have, for example,

$$\sum_{u=1}^{m} [\sqrt{\hat{q}_u}]^2 = \hat{q}^\top \hat{q} = \sum_{u=1}^{m} \hat{q}_u = 1.$$ 

The distance $d(\hat{p}, \hat{q}) = \sqrt{1 - \rho(\hat{p}, \hat{q})}$ is a metric.

The Jensen inequality, for a scalar concave(!) function $f(x)$ ($f''(x) < 0$), is

$$\mathbb{E}[f(x)] \leq f(\mathbb{E}[x])$$

For example, we have the numbers $x_1, x_2, \ldots, x_m, x_i \geq 0$. Since $ln(x)$ is concave, $(ln(x))'' = -1/x^2$,

$$\frac{ln(x_1) + ln(x_2) + \cdots + ln(x_m)}{m} \leq ln\left(\frac{x_1 + x_2 + \cdots + x_m}{m}\right)$$

or by eliminating the logarithm

$$\sqrt[m]{x_1 x_2 \cdots x_m} \leq \frac{x_1 + x_2 + \cdots + x_m}{m}$$

with equality iff $x_1 = x_2 = \cdots = x_m$. 
In our case

\[
\rho(\hat{p}, \hat{q}) = \sum_{i=1}^{m} \hat{p}_u \sqrt{\frac{\hat{q}_u}{\hat{p}_u}} \leq \sqrt{\sum_{i=1}^{m} \hat{p}_u \frac{\hat{q}_u}{\hat{p}_u}} = 1
\]

where \( f(x) = \sqrt{x} \), a concave function \( f''(x) = -1/(4x^{1.5}) \) and \( x = \frac{\hat{q}_u}{\hat{p}_u} \).

Therefore the distance \( d(\hat{p}, \hat{q}) \) is between 0 and 1, symmetric and equal to zero when \( \hat{p} = \hat{q} \).

The triangle inequality to be satisfied is

\[
d(\hat{p}, \hat{r}) + d(\hat{r}, \hat{q}) \geq d(\hat{p}, \hat{q})
\]

Vectors \( \hat{p}, \hat{r}, \hat{q} \) are points on the unit hypersphere: \( \xi_p, \xi_r, \xi_q \). This is equivalent to

\[
\sqrt{1 - \cos(\xi_p, \xi_r)} + \sqrt{1 - \cos(\xi_r, \xi_q)} \geq \sqrt{1 - \cos(\xi_q, \xi_q)}.
\]

The three cosine functions can be written as

\[
\left| \sin \left(\frac{\xi_p + \xi_r}{2}\right) \right| + \left| \sin \left(\frac{\xi_r + \xi_q}{2}\right) \right| \geq \left| \sin \left(\frac{\xi_p + \xi_q}{2}\right) \right|.
\]

If we fix two of the points \( \xi_p \) and \( \xi_q \), the left side of the inequality is minimized when all three points are in the plane \( < \xi_p, \xi_q > \) going through the origin and therefore intersecting the hypersphere. Therefore, the three points on the hypersphere always satisfy the triangle inequality, even if they are on an intersecting plane.