Planar Homography.

Let the affine coordinates be for a point in a plane in the first images \((x_1 \ x_2 \ 1)\), and for a point in a plane the second image \((m_1 \ m_2 \ 1)\). A 2D planar homography is a projective transformation between the corresponding points in the two planes

\[
\begin{pmatrix} m \\ 1 \end{pmatrix} \sim H \begin{pmatrix} x \\ 1 \end{pmatrix}
\]

where \(H\) is a \(3 \times 3\) matrix which eight degrees of freedom. The projective equivalence, \(\sim\), means that the second image point is \((w_1 \ w_2 \ w_3)\), which is equivalent to \(m = (w_1/w_3 \ w_2/w_3 \ 1)\).

Assume that \(h_{33} \neq 0\). Another \(h_{ij}\) can be assumed if \(h_{33} = 0\). We have only eight unknowns.

\[
H = \begin{pmatrix} A & t \\ v^\top & 1 \end{pmatrix}
\]

where \(A\) is a \(2 \times 2\) affine matrix, \(v\) is the projective deformation vector and \(t\) is a vector which is proportional but not equivalent with the translation.

If you map from a neighborhood center on \(x_0\) to the neighborhood center on \(m_0\)

\[
\begin{pmatrix} m - m_0 \\ 1 \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ v^\top & 1 \end{pmatrix} \begin{pmatrix} x - x_0 \\ 1 \end{pmatrix}
\]

then \(t = -Ax_0\). Locally, you map a small plane between the two images.

Any \(2 \times 2\) affine matrix \(A\) can be written based on svd

\[
A = USV^\top = (UV^\top)(VSV^\top) = R_0 \ R_1^\top S R_1
\]

where \(S = \text{diag}(s_1, s_2)\). At the beginning there is rotation with \(R_1\) with rotates with \(\alpha_1\). The point is anisotropically scaled with \(s_1\) and \(s_2\). After that the rotation with \(R_1^\top\) rotates back. The angle \(\alpha_1\) is used only as the scaling direction. Finally \(R_0\) sets the angle of the transformation.

We have four unknowns for the affine matrix \(A\), two unknowns for the projective deformation vector \(v\), and two for spatial coordinates of \(m_0\), a total of eight unknowns. The neighborhood center \(x_0\) is known.
Let a matrix be partitioned into a block form:

\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

The blocks \( A \) \((m \times m)\) and \( D \) \((n \times n)\) are invertible. The product \( MM^{-1} = M^{-1}M = I_{m+n} \).

\[
M^{-1} = \begin{pmatrix}
(A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1}
\end{pmatrix}
\]

We check the formula for a \( 2 \times 2 \) matrix

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d(ad - cb)^{-1} & -b(ad - cb)^{-1} \\ -c(ad - cb)^{-1} & a(ad - cb)^{-1} \end{pmatrix}
\]

If we apply matrix inversion to a projective matrix without translation

\[
H = \begin{pmatrix} A & 0 \\ \mathbf{v}^\top & 1 \end{pmatrix} \quad H^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -\mathbf{v}^\top A^{-1} & 1 \end{pmatrix}
\]

where \( A^{-1} = R_1^T S^{-1} R_1 R_0^\top \) since is composed from \( 2 \times 2 \) square matrices.

The relation from the second image to the first image is

\[
\begin{pmatrix} x - x_0 \\ 1 \end{pmatrix} \sim \begin{pmatrix} A^{-1} & 0 \\ -\mathbf{v}^\top A^{-1} & 1 \end{pmatrix} \begin{pmatrix} m - m_0 \\ 1 \end{pmatrix}
\]

and have to find the 8-dimensional vector \( \theta \): the coordinates \( m_{0x}, m_{0y} \), the angles \( \alpha_0, \alpha_1 \), the scales \( s_x, s_y \) and the projective deformation \( v_0, v_1 \). Note that \( x_0 \) is given.