Algebraic distance.

A surface (or curve) in $\mathbb{R}^q$ is defined implicitly as $f(y_o) = 0$, where $y_o$ is a point on the surface and $f(\cdot)$ is a scalar. If we take a point $y$ not on the surface, the value $f(y) \neq 0$. The sign can be negative or positive function of the point is ”inside” or ”outside” the surface.

From $y$ we can draw the Euclidean distance to the surface. The distance gives the shortest path (in $L_2$) and hits the surface in $\hat{y}$. So $f(\hat{y}) = 0$. The first order Taylor series of $f(y)$ around $\hat{y}$

$$f(y) \simeq f(\hat{y}) + \nabla_y f(\hat{y})^\top (y - \hat{y}) = \nabla_y f(\hat{y})^\top (y - \hat{y})$$

which is acceptable if the surface is not very nonlinear. If the surface is linear, like in many estimation problems, the above relation is exact. In this case, the gradient in $\hat{y}$ in along the direction $y - \hat{y}$.

Taking the norms in $L_2$

$$|f(y)|_2 \simeq ||\nabla_y f(\hat{y})||_2 ||y - \hat{y}||_2$$

based on the Cauchy-Schwartz inequality $|(a, b)|_2 \leq ||a||_2 ||b||_2$.

Algebraic distance is

$$||y - \hat{y}||_2 \simeq \frac{|f(y)|_2}{||\nabla_y f(\hat{y})||_2} \simeq \frac{|f(y)|_2}{||\nabla_y f(y)||_2}$$

where we usually substitute $\hat{y}$ with $y$ to have the approximation.

In our linear case

$$g(y_o) = y_{io}^\top \theta - \alpha = 0 \quad ||\theta|| = 1 \quad \alpha \geq 0$$

with $i = 1, \ldots, n$. The gradient is constant

$$\nabla_y g(\hat{y}) = \nabla_y g(y) = \theta$$

and the norm of the gradient is one. Therefore $||y - \hat{y}|| = |g(y)|$. 