CONTROL OF THE ELASTIC MOTION OF LIGHTWEIGHT STRUCTURES

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Abstract

We apply multivariable control techniques to control the elastic motion of lightweight multi-body structures undergoing combined rigid and elastic motion. The elastic control action is implemented by means of segmented piezoelectric film actuators. We show that, by using piezoelectric actuators, one can accomplish input decoupling as well as true internal actuation. We also show that the overall control design can be broken into individual components, with each component resulting in a stable controller. This makes it much simpler to design and implement the control action.

Introduction

In this paper, we apply multivariable control techniques to control the elastic motion of lightweight structures. It is expected that many of the space structures considered for future space applications will be highly flexible, so that at least part of the elastic motion will require active control. While one can expect that the high frequency modes will be damped out by existing internal damping or by passive damping enhancements to the structure, in many cases the lower frequency modes will require active damping.

In past years, multivariable control of elastic motion with several sensors and actuators was considered a theoretically possible but practically infeasible approach. This was because the sensors and actuators that were available then were so massive that they would add substantially to the weight of the structure to be controlled. Also, actuator dynamics changed the mathematical model of the structure quite a bit.

Recent developments in piezoelectric materials have made it possible to place several sensors and actuators on structures, without altering their weight or mathematical model [e.g., 1-5]. Especially, use of thin film sensors and actuators are extremely desirable for structures with large surface areas, such as solar panels or shells. Also, because these sensors and actuators can

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be cut into any size and shape, previous restrictions on the number of sensors and actuators are basically eliminated.

One advantage of using piezoelectric film sensors is that they measure strain. We show in this paper that this property is particularly advantageous in structures exhibiting both rigid body as well as elastic motion. The rigid body motion does not affect the piezoelectric sensor’s measurement. Also, we show in this paper that a piezoelectric layer actuator permits input decoupling and internal actuation. Hence, piezoelectric components have an advantage over traditional actuators.

We propose to use a two-tier control approach. One set of controls accomplishes the rigid body control, such as a slew maneuver or rotation of solar panels. A second set of controls, in the form of piezoelectric film sensors and actuators, annihilates the elastic motion. While accomplishing the rigid body control the first set of actuators end up directly exciting the elastic motion, the second set of actuators that annihilate the elastic motion do not contribute to terms to the describing equations for the rigid body motion. As a result, the elastic motion control can be designed without considering the rigid body motion, and a stable result is guaranteed when the mathematical model is correct.

A disadvantage of piezoelectric film actuators is that they can only impart limited amounts of control effort. For example, controlling a thick plate by piezoelectric film actuators is very difficult. On the other hand, because future spacecraft and components are expected to be lightweight, we anticipate that piezoelectric film actuators will be ideally suited for their control.

The Piezoelectric Effect

Consider a plate or a beam to which a piezoelectric thin film is attached, as shown in Fig. 1. This film can be used as a sensor or an actuator. The equation of motion can be expressed as

\[ D_E \nabla^4 w(P, t) + \rho h \frac{\partial^2 w(P, t)}{\partial t^2} = f(P, t) \]  

in which \( \nabla^4 \) is the biharmonic operator, \( w \) is the displacement in the transverse direction, \( P \) is the spatial variable, \( x \) for beams, \( x, y \) for plates.

Let us now consider that the measurement and/or control actions are accomplished by thin piezoelectric films placed on the structure. These elements have the property of being extremely thin so that the mass and stiffness they add to the structure is negligible. Beginning with sensors, and placing \( m_s \) sensors on the structure, the charge measured by the \( j \)-th sensor has the form [3]

\[ q_j = -\bar{z}_j \int_0^a \int_0^b \left\{ F_j^s(x, y) P_0 L_w[w(x, y, t)] \right\} dxdy \quad j = 1, 2, \ldots, m_s \]

where \( q_j \) is the charge measured by the \( j \)-th piezoelectric film, \( \bar{z}_j \) is the distance in the \( z \)-direction from the center of the plate to the center of the piezoelectric film, \( P_0 = P_0(x, y) \) denotes the
polarity of the film, $F_j^s(x, y)$ spatial coverage operator for the $j$-th sensor, so that it is equal to 1 if $(x, y)$ is within the electrode coverage and 0 otherwise, and $L_w$ is the charge operator having the form

$$L_w = e_{31} \frac{\partial^2}{\partial x^2} + e_{32} \frac{\partial^2}{\partial y^2} + 2e_{36} \frac{\partial^2}{\partial x\partial y}$$

with $e_{31}$, $e_{32}$ and $e_{36}$ the piezoelectric constants of the material. In our study, none of the sensors and actuators are stacked and we consider that the thickness of the piezoelectric film is much smaller than the thickness of the plate, so that $\bar{z}_j \approx h/2$, ($j = 1, 2, \cdots m_s$), with $h$ denoting the thickness of the plate.

When piezoelectric films are used as actuators, the applied electric field can be expressed as the product of the spatial pattern of the effective surface electrode and a time signal as

$$E_{3,k}(x, y, t) = G_k(t)F_k^a(x, y), \ k = 1, 2, \cdots, m_a$$

where $G_k(t)$ is the electric field applied to the $k$-th actuator, $F_k^a(x, y)$ is the spatial coverage operator discussed earlier and $m_a$ is the number of actuators.

It follows that the right side of Eq. (1) can be expressed as

$$f(x, y, t) = \sum_{k=1}^{m_a} h_k \bar{z}_k L_w[E_{3,k}(x, y, t)]$$

where $h_k$ is the thickness of the $k$-th piezo actuator and $L_w$ was defined in Eq. (3). In our case, we are considering actuators of the same thickness, so we replace $h_k$ ($k = 1, 2, \cdots, m_a$) with $h_a$, denoting the thickness of all actuators. Note the similarity of the actuator and sensor equations.

We next obtain the modal equations of motion. We use the standard modal expansions of

$$w(x, y, t) = \sum_{s=1}^{\infty} \phi_s(x, y)\eta_s(t)$$

for a plate and

$$w(x, t) = \sum_{s=1}^{\infty} \phi_s(x)\eta_s(t)$$

for a beam, where $\phi_s(x, y)$ and $\phi_s(x)$ are corresponding eigenfunctions. We next define the coverage area of the actuators and sensors. We assume that all strips have the same polarity $P_b(x, y) = 1$ and that the piezoelectric constants are the same for all actuators and the same for all the sensors. For a beam, the $k$-th actuator strip begins at $x = x_k^a$ and is of length $a_k^a$ and width $b_k^a$. For a plate, the $k$-th strip begins at $(x_k^s, y_k^s)$ and is of length $a_k^s$ and width $b_k^s$. We use the same notation with superscript $s$ for the sensors.

Introducing the above expansions to the equations of motion and invoking orthogonality conditions we obtain the modal equations of motion in the form

$$\{\ddot{\eta}(t)\} + [\Lambda]\{\eta(t)\} = [B^a]^T\{h_a G(t)\}$$

where $B^a$
in which $[\Lambda]$ is a diagonal matrix containing the eigenvalues, $\{G(t)\} = [G_1(t) \ G_2(t) \ \cdots \ G_{ma}(t)]^T$ is a vector of applied electric fields, and $[B^a]$ is the control influence matrix, whose entries have the form \[\begin{align*}
B_{ks}^a &= -\frac{h}{2} e_{31} \int y_{k}^{a+k} \left( \frac{\partial \phi_s(x,y)}{\partial x} \right) \, dy - \frac{h}{2} e_{32} \int x_{k}^{a+k} \left( \frac{\partial \phi_s(x,y)}{\partial y} \right) \, dx \\
&\quad - 2\frac{h}{2} e_{36} \left( \phi_s(x,y) \right) x_{k}^{a+k} \left( \frac{\partial \phi_s(x,y)}{\partial x} \right) \, dy \\
&\quad - 2\frac{h}{2} e_{36} \left( \frac{\partial \phi_s(x,y)}{\partial y} \right) x_{k}^{a+k} \, dx \\
&= \begin{bmatrix} \frac{d\phi_s(x)}{dx} \bigg|_{x=x_k} - \frac{d\phi_s(x)}{dx} \bigg|_{x=x_k} \end{bmatrix} \ k = 1, 2, \cdots, ma, \ s = 1, 2, \cdots, \infty
\end{align*}\] for a plate and
\[\begin{align*}
B_{ks}^a &= -e_{31} \frac{h}{2} b_k \left[ \frac{d\phi_s(x)}{dx} \bigg|_{x=x_k} - \frac{d\phi_s(x)}{dx} \bigg|_{x=x_k} \right] \ k = 1, 2, \cdots, ma, \ s = 1, 2, \cdots, \infty
\end{align*}\] for a beam. For other types of actuators, one can show that the modal equations of motion will have the same form as Eq. (8), with the entries of $[B^a]$ having different forms than the above equations.

It follows that the expressions that relate the charge to the modal coordinates has the form as \[\{q(t)\} = [B^s]\{\eta(t)\}\] for both the plate and the beam, where $\{q(t)\} = [q_1(t) \ q_2(t) \ \cdots \ q_{ms}(t)]^T$ contains the measured charges. The sensor influence matrix $[B^s]$ has the same form as the actuator influence matrix $[B^a]$, with the superscript $a$ replaced by $s$, and $ma$ replaced by $ms$.

**Rigid Body Modes and Control**

We are interested in controlling the motion of elastic bodies that admit rigid body motion. We propose the following strategy for accomplishing the combined rigid body and elastic motion control. One set of actuators accomplish the rigid body motion control, and another set control the elastic motion. We justify this approach by noting that a) the rigid body motion control requires larger control inputs than the elastic motion control, b) in many cases the rigid body motion control is accomplished by open-loop control and elastic motion by feedback, and c) usually, rigid body motion control is accomplished by different types of actuators. For example, in spin stabilized spacecraft, attitude control is accomplished by rotors. While combined rigid body and elastic motion control is a feasible approach for small order systems, for larger order systems the control design becomes difficult.

The disadvantage associated with using two sets of actuators and two sets of control laws is that a control action designed to control the rigid body motion will influence the elastic motion and control of the elastic motion will excite the rigid body motion. Hence, an additional objective of the control design is to minimize the effects of one level of control on the other. It
is in achieving this effect that using piezoelectric film elements as sensing and control devices comes in very handy.

Let us examine the elements of the control and sensing influence matrices. In general, and when discrete sensors and actuators are used, these matrices are fully populated. However, the situation is different with systems admitting rigid body motion. Consider first small motions (both rigid body and elastic), which leads to a set of linear equations. For a free-free beam, there are two rigid body modes, and they can be expressed as translational and rotational

\[ \phi_1(x) = A_1 \quad \phi_2(x) = A_2(x - a/2) \]

in which \( A_1 \) and \( A_2 \) are amplitude constants, usually determined by making use of orthogonality. Considering the definition of \( B_{rs} \) from Eq. (10)

\[ B_{r1} = 0 \quad B_{r2} = 0 \quad r = 1, 2, \cdots, m_s \]  

(13)

Not surprisingly, we get the same result for a plate, or for that matter a shell, or any other structure exhibiting rigid body motion. Indeed, for a free-free rectangular plate, the rigid body modes associated with the motion in the \( z \) direction have the form

\[ \phi_1(x, y) = D_1 \quad \phi_2(x, y) = D_2(x - a/2) \quad \phi_3(x, y) = D_3(y - b/2) \]

in which \( D_1, D_2 \) and \( D_3 \) are amplitude constants. It follows from Eq. (9) that Eq. (13) is valid for plates as well (for a free-free plate, \( B_{r3} = 0 \) as well).

Let us next observe the implications of the above result. Consider the sensing equation first and, without loss of generality, a structure that has one rigid body mode, so that \( B_{r1} = 0, \, (r = 1, 2, \cdots, m_s) \). We express Eq. (11) as

\[
\begin{bmatrix}
q_1(t) \\
q_2(t) \\
\vdots \\
q_{m_s}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & B_{12} & B_{13} & \cdots \\
0 & B_{22} & B_{23} & \cdots \\
\vdots & \vdots & \vdots & \cdots \\
0 & B_{m_s2} & B_{m_s3} & \cdots
\end{bmatrix}
\begin{bmatrix}
\eta_1(t) \\
\eta_2(t) \\
\eta_3(t) \\
\vdots
\end{bmatrix}
\]

(15)

We see that the rigid body mode does not affect the piezoelectric sensors output. This, of course is an expected result, because piezoelectric materials only emit electrical signals when they are deformed and there is no deformation associated with rigid body modes. We will see the significance of this result when we discuss multibody systems later on.

Let us now consider the actuator equations in modal form. Again, considering one rigid body mode, we can write Eq. (8) as

\[
\begin{bmatrix}
\ddot{\eta}_1(t) \\
\ddot{\eta}_2(t) + \omega_2^2 \eta_2(t) \\
\ddot{\eta}_3(t) + \omega_3^2 \eta_3(t) \\
\vdots
\end{bmatrix} = h_a
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
B_{12} & B_{13} & \cdots & B_{m_s2} \\
B_{13} & B_{23} & \cdots & B_{m_s3} \\
\vdots & \vdots & \cdots & \vdots
\end{bmatrix}
\begin{bmatrix}
G_1(t) \\
G_2(t) \\
G_3(t) \\
\vdots
\end{bmatrix}
\]

(16)
We see from the above equation that for small motions the piezoelectric film actuators do not excite the rigid body mode. This result, while expected from the sensors result, indicates that no matter what control law we select to control the elastic motion, the elastic motion control does not affect the rigid body motion.

When large angle rigid body motion is considered, we obtain results similar to the above. Let us first consider a large angle motion problem, such as a pinned-free beam undergoing large angle rigid body motion and elastic motion, as shown in Fig. 2.

We view the elastic motion from a moving reference frame [6] - [7]. Motion of reference frame is referred to as the primary motion and motion viewed from the reference frame is the secondary motion [6]. The reference frame should be selected so that the secondary motion is small and it can be modeled using linear theory.

We write the position of a point on the beam as

$$\mathbf{r}(x, t) = \mathbf{r}_O(t) + x\hat{i} + u(x, t)\hat{i} + v(x, t)\hat{j} + w(x, t)\hat{k}$$  (17)

As can be seen from Fig. 2, one has several choices of selecting the reference frame. In the following, we summarize three options.

Selection of reference frame changes boundary conditions on the secondary motion and introduces a constraint on the secondary motion and changes its boundary conditions. Note that this constraint is only mathematical. We consider here deformation in y-direction only ($v(x, t)$) and identify three commonly used approaches:

1) Zero Slope Constraint ($\theta''$, $x''y''$): Here, the reference frame is selected such that it is aligned with the slope of the beam. The constraint that is applied is $v'(0, t) = 0$. Hence, the boundary conditions of the secondary motion become that of a fixed-free beam.

2) Rigid Body Mode Constraint ($\theta'$, $x'y'$): Here, the reference frame is selected such that the secondary motion has no rigid body component, $\int_0^L \rho(x')x'v(x', t)dx' = 0$. This reference frame does not alter the boundary conditions of the secondary motion. This feature is attractive from a mathematical view. However, issues arising from multibody mechanics and difficulties associated with measuring the orientation of this frame reduce its desirability.

3) Zero Tip Deformation Constraint ($\theta$, $xy$): Here, the reference frame is selected by drawing a line from the pin joint to the tip of the beam. Hence, $v(L, t) = 0$ and the secondary motion has the boundary conditions of a pinned-pinned beam. This last selection is more desirable than the previous two because of the ease with which the reference frame can be located, the simplicity of trial functions to be used to express the secondary motion (sinusoidals), numerical stability, and physical consistency when dealing with multibody systems. We will use this reference frame in the remainder of this paper.

For modal analysis, one expresses the secondary motion as

$$v(x, t) = \sum_{s=1}^n \phi_s(x)\eta_s(t)$$  (18)
where $\phi_s(x)$ are trial functions and $\eta_s(t)$ are generalized coordinates. We next derive the equations of motion. As discussed earlier, the control approach here is to use separate sets of actuators to accomplish the rigid body and elastic motion control. For the single link at hand, we use a single actuator to exert a moment $M(t)$ at the pin joint and a series of piezoelectric actuators at different points along the link. Using a Lagrangian approach, the virtual work can be written as

$$
\delta W = M(\delta \theta + \delta v'(0)) + \int f(x) \cdot \delta r(x) dx = M(\delta \theta + \delta v'(0))
$$

$$
+ \int \sum_{k=1}^{m_a} h_k \tilde{z}_k L_w[\mathcal{E}_{3,k}(x)] \cdot \delta r(x) dx
$$

(19)

Let us consider the variation of $r(x)$. Because $r(x,t) = (x + u(x,t))i + v(x,t)j$ (with $u(x,t)$ denoting the shortening of the projection, which is assumed to be very small) we can write this variation as

$$
\delta r(x) = \delta(x + u)i + \delta v(x)j + (x + u)\delta i + v(x)\delta j
$$

$$
\approx \delta x i + \delta u i + \delta v(x)j + \delta \theta x j - \delta \theta v(x)i
$$

(20)

so that the virtual work becomes

$$
\delta W = M(\delta \theta + \delta v'(0)) + \int \sum_{k=1}^{m_a} h_k \tilde{z}_k L_w[\mathcal{E}_{3,k}(x)](\delta v(x) + x\delta \theta) dx
$$

(21)

Note that by definition of $\mathcal{E}_{3,k}(x)$, the area of application of each piezoelectric actuator is defined. Introducing the modal expansion to the above equation we obtain

$$
\delta W = M\delta \theta + M \sum_{n=1}^{n} \phi'_s(0)\delta \eta_n + \sum_{k=1}^{m_a} \sum_{n=1}^{n} h_a G_k B_k \delta \eta_n
$$

$$
= M\delta \theta + M\{\phi'(0)\}^T\{\delta \eta\} + \{h_a G\}^T [B]\{\delta \eta\}
$$

(22)

Note that the last term in the integrand of Eq. (21), $x\delta \theta$, does not contribute to the virtual work, so that the piezoelectric actuators will not lead to a term on the right side of the equation of motion associated with $\theta$. This property is referred to as input decoupling. Furthermore, we see that regardless of which way we select the moving reference frame and the trial functions, we get the same result.

Using a standard Lagrangian approach, one can show that the equations of motion have the form [6], [8]

$$
\frac{d}{dt} \left( [I_0 + \sum_{r=1}^{n} \sum_{s=1}^{n} m_{rs} \eta_r \eta_s] \dot{\theta} \right) + \sum_{r=1}^{n} a_r \ddot{\eta}_r = M
$$

$$
a_s \ddot{\theta} + \sum_{r=1}^{n} (m_{sr} \ddot{\eta}_r + \dot{\theta}^2 (h_{sr} - m_{sr}) + k_{sr}) \eta_r = M\phi_s'(0) + \sum_{k=1}^{m_a} B_k h_a G_k \quad s = 1, 2, \cdots
$$

(23)
where $B_{ks}$ are defined the same way as in Eq. (10). The piezo actuators do not appear in the equation of motion for the primary motion. Of course, the primary and secondary motions are coupled, which means that the piezoelectric actuators end up affecting the rigid body motion, but the above equation suggests that one can design the secondary motion control without regard to the primary motion. This, in essence implies that the piezoelectric film actuators do not affect the angular momentum about the pin joint.

For other types of actuators, the $x\delta \theta$ term in Eq. (21) does contribute to the virtual work and hence one obtains the result that the elastic motion control leads to terms on the right side of the primary motion. Hence, there would be no input decoupling. One can design the control to prevent this from happening, by means of imposing a constraint on the control law. This complicates the control design.

It follows that the sensor equation has the same form as Eq. (11), so that one can use any approach to locate the reference frame associated with the primary motion.

We obtain the same result for multibody systems. Figure 3 shows a double link with both components being elastic, using the zero tip deformation constraint. The position of points on the first and second links can be written as

$$r(x_1, t) = x_1 i_1 + v_1(x_1, t) j_1 \quad r(x_2, t) = L_1 i_1 + v_1(L_1, t) j_1 + x_2 i_2 + v_2(x_2, t) j_2$$  \hspace{1cm} (24)

Here, we only concentrate on the expression for virtual work and the right hand side of the equations of motion. Noting from the above example that the shortening of the projection does not have an effect on the virtual work, we ignore it here for the purpose of writing the virtual work. We expand the secondary motion using

$$v_1(x_1, t) = \sum_{s=1}^{n_1} \phi_s(x_1) \eta_s(t) \quad v_2(x_2, t) = \sum_{s=1}^{n_2} \psi_s(x_2) \xi_s(t)$$  \hspace{1cm} (25)

and consider that the control action is implemented by two torquers $M_1$ and $M_2$ on the pin joints and two sets of piezoelectric film actuators. Using the expansions above, we can write the variations of the individual links as

$$\delta r_1 = \delta x_1 i_1 + \sum_{s=1}^{n_1} \phi_s(x_1) \delta \eta_s j_1 + \delta \theta_1 x_1 j_1 - \delta \theta_1 \sum_{s=1}^{n_1} \phi_s(x_1) \eta_s i_1$$  \hspace{1cm} (26)

$$\delta r_2 = \sum_{s=1}^{n_1} \phi_s(L_1) \delta \eta_s j_1 + \delta \theta_1 L_1 j_1 - \delta \theta_1 \sum_{s=1}^{n_1} \phi_s(L_1) \eta_s i_1$$

$$\quad + \delta x_2 i_2 + \sum_{s=1}^{n_2} \psi_s(x_2) \delta \xi_s j_2 + \delta \theta_2 x_2 j_2 - \delta \theta_2 \sum_{s=1}^{n_2} \psi_s(x_1) \xi_s i_2$$  \hspace{1cm} (27)

Note that when the zero tip deformation constraint is used, $\phi_s(L_1) = 0$ and considerable simplification of the above equation is achieved. This is yet another advantage of the zero tip deformation constraint approach. We write the virtual work as

$$\delta W = \delta W_1 + \delta W_2$$  \hspace{1cm} (28)
where

\[
\delta W_1 = M_1 (\delta \theta_1 + \delta v'_1(0)) + \int \sum_{k=1}^{m_0} h_k \ddot{z}_k L_w [E_{3,k}(x_1)] j_1 \cdot \delta r_1(x_1) dx_1
\]

\[
\delta W_2 = M_2 (\delta \theta_2 + \delta v'_2(0)) + \int \sum_{k=1}^{m_0} h_k \ddot{z}_k L_w [E_{3,k}(x_2)] j_2 \cdot \delta r_2(x_2) dx_2
\]

with the subscripts denoting the links. For the virtual work of the external inputs acting on the first link, we get the same expression as Eq. (22),

\[
\delta W_1 = M_1 \delta \theta_1 + M_1 \{\phi'(0)\}^T \{\delta \eta\} + \{h_a G_1\}^T [B_1] \{\delta \eta\}
\]

(30)

Next, we examine the virtual work associated with the second link. The first three terms do not contribute to the virtual work, as any integration over \(\delta x_2\) does not treat them as variables. The last four terms in Eq. (27) give the same result as the above equation

\[
\delta W_2 = M_2 \delta \theta_2 + M_2 \{\psi'(0)\}^T \{\delta \xi\} + \{h_a G_2\}^T [B_2] \{\delta \xi\}
\]

(31)

Hence, the piezoelectric actuators of one link do not provide a term for the right sides of the equations of motion associated with the primary motion of either link. Further, they do not provide control terms on the right sides of the describing equations for the secondary motion of the other link. This is a powerful result, which suggests that one can design the secondary motion control of each link independently of the other links, and independently of the rigid body motion control. This property is known as internal actuation ([9] - [11]), where a control law is designed using excitation sources internal to the system. In general, using a control law to obtain internal actuation leads to very complex control laws and limited to low-order models. In our case, however, internal actuation is achieved as a result of the system hardware, that is, by means of the piezoelectric film actuators. The property of internal actuation, where a separate and stable vibration controller can be designed for each link independently of the others, makes the control design process much simpler. Also, it is clear from the above derivation that the results obtained for a double link can be extended to any number of linkages.

Finally, consider the spacecraft shown in Fig. 4 with a rigid hub and flexible panels. We use a set of local axes \(xyz\) attached to the spacecraft and another set \(x'y'z'\) attached to the undeformed position of the solar panel. The second coordinate set is obtained by obtaining the first set by an angle \(\theta_1\) about the \(x-\)axis.

Let us assume that the deformation of a point on the solar panel is small and along the \(z'\)-axis only. The position of a point on the solar panel is then written as

\[
r(x, y', t) = r_H + Li + w(x, y', t)k'
\]

(32)

in which \(r_H\) denotes the position of the hub and \(w(x, y', t)k'\) is the elastic deformation of the panel. Let us assume that the hub is rotating with angular velocity \(\Omega k\). The velocity of this
point becomes
\[ \dot{\mathbf{r}}(x, y', t) = \dot{\mathbf{r}}_H + \mathbf{\Omega} \times \mathbf{L} \mathbf{i} + (\mathbf{\Omega} \mathbf{k} + \dot{\mathbf{\theta}}_1 \mathbf{i}) \times w(x, y', t) \mathbf{k}' + \dot{w}(x, y', t) \mathbf{k}' \] (33)

The equations of motion can be obtained by plugging the expression for the velocity into the kinetic energy, developing the expression for the potential energy and using Lagrange’s equations [6]. Here, we again consider the virtual work by piezoelectric actuators placed on the solar panels and write it as
\[ \delta W = \int f(x, y') \mathbf{k}' \cdot \delta \mathbf{r}(x, y') dx dy' = \int \sum_{k=1}^{m_a} h_k \tilde{z}_k L_w [\mathbf{E}_3, k(x, y, t)] \mathbf{k}' \cdot \delta \mathbf{r}(x, y') dx dy' \] (34)

Recall that the virtual displacement \( \delta \mathbf{r} \) can be obtained directly from Eq. (33) by replacing the time derivative with variation. We can also consider \( \mathbf{\Omega} \) as a variable by writing it as \( \dot{\psi} \). Considering the right side of Eq. (33), the virtual work due to the first two terms vanishes because these terms are constants when integrating w.r.t. \( x \) and \( y' \) so they vanish when integration by parts is used. The third term, \( (\mathbf{\Omega} \mathbf{k} + \dot{\mathbf{\theta}}_1 \mathbf{i}) \times w(x, y', t) \mathbf{k}' \), does not contribute because it has no component in the \( z' \) direction. Hence, the only term that contributes to the virtual work is the last one, and we get the same result as if the solar panel was a simple plate.

We conclude from the above that in a multibody environment, using piezoelectric sensors and actuators have the desirable attribute that they only impart terms to the equations of motion associated with the body they are attached to. They do not enter the describing equations of rigid body motion and the elastic motion of the other bodies. Of course, these motions are related to each other through terms on the left side of the equations of motion, but the mere fact that internal actuation is possible hints that we can take advantage of this and decentralize the control design. We pursue this idea in the next section.

**Control Design**

Recall the double link considered earlier. We use \( n_1 \) and \( n_2 \) trial functions to model each link and we write the generalized coordinate vector as
\[ \{x\} = [\theta_1 \theta_2 \eta_1 \eta_2 \cdots \eta_{n_1} \xi_1 \xi_2 \cdots \xi_{n_2}]^T \] (35)
or in array form
\[ \{x\} = [\{\theta\}^T \{\eta\}^T \{\xi\}^T]^T \] (36)
and we write the force vector as \( \{F\} = \{F_R\} + \{F_E\} \), with the subscripts \( R \) and \( E \) denoting rigid and elastic, respectively, and where
\[ \{F_R\} = [\mathbf{M}_1 \mathbf{M}_2 0 0 \cdots 0 0 0 \cdots 0]^T \]
\{F_E\} = [0 \ 0 \ h_aG_1^1 \ h_aG_2^1 \ \cdots \ h_aG_{m_1}^1 \ h_aG_1^2 \ h_aG_2^2 \ \cdots \ h_aG_{m_2}^2]^T \quad (37)

or in array form

\{F_R\} = \left[\{M\}^T \ \{0\}^T \ \{0\}\right]^T \quad \{F_E\} = \left[\{0\}^T \ \{h_aG_1\}^T \ \{h_aG_2\}^T\right]^T \quad (38)

so that the equations of motion can be written in the generalized form

\[ [M]\{\ddot{x}\} + \{g(\{x\}, \{\dot{x}\})\} = [D_R] \{F_R\} + [D_E] \{F_E\} \quad (39) \]

in which \([M]\) is a fully populated inertia matrix

\[
[M] = \begin{bmatrix} [M_{11}] & [M_{12}] & [M_{13}] \\ [M_{21}] & [M_{22}] & [M_{23}] \\ [M_{31}] & [M_{32}] & [M_{33}] \end{bmatrix} \quad (40)
\]

\([g(\{\eta\}, \{\dot{\eta}\})]\) is a vector containing coriolis, centrifugal, stiffness and other coupling terms, and \([D_R]\) and \([D_E]\) are the control influence matrices

\[
[D_R] = \begin{bmatrix} [D_{11}] & [0] & [0] \\ [D_{21}] & [0] & [0] \\ [D_{31}] & [0] & [0] \end{bmatrix} \quad [D_E] = \begin{bmatrix} [0] & [0] & [0] \\ [0] & [B_1]^T & [0] \\ [0] & [0] & [B_2]^T \end{bmatrix} \quad (41)
\]

Note that the matrices \([B_1]\) and \([B_2]\) in the above equation were introduced in in Eqs. (30)-(31). When we use the zero tip deformation constraint, one can show that \([M_{23}] = [0], [M_{32}] = [0]\).

The control of the primary motion can be carried out in a number of ways, including feedforward and feedback, and is outside the interest of this paper. When designing the primary motion using feedforward control, one should tailor the control input so as to excite the secondary motion as little as possible.

We are interested in designing the secondary motion control law, so we consider \(\{F_E\}\) and \([D_E]\). Hence, the contribution to the right side of the equations of motion are

\[
\{F_E\} = \begin{bmatrix} \{0\} \\ [B_1]^T \{h_aG_1\} \\ [B_2]^T \{h_aG_2\} \end{bmatrix} \quad (42)
\]

We design the control law by

\[
[B_1]^T \{h_aG_1\} = -[C_1]\{\dot{\eta}\} \quad [B_2]^T \{h_aG_2\} = -[C_2]\{\xi\} \quad (43)
\]

in which \([C_1]\) and \([C_2]\) are positive semidefinite gain matrices. A simple way of selecting them is as diagonal. These entries can be constant or they can be continuously adjustable using a control criteria. Obviously, the positive semidefiniteness of the gain matrices guarantees stability, provided a sufficient number of terms are used to expand the secondary motion and the time derivatives are estimated accurately from the charge measurements. Also, because of the dynamic interactions between the primary and secondary motions, the stability property positively influences the primary motions, as well.
Note that the control law above requires the velocities associated with the terms in the expansion of the secondary motion. The piezoelectric film sensors provide only coordinates, so that an observation scheme needs to be employed to calculate the velocities when piezoelectric film sensors are used. Such an observation scheme has been developed in [5].

We select the number of actuators for each link the same as the number of terms in the expansion of the secondary motion. This is a relatively easy requirement to satisfy, due to segmentation of the piezoelectric actuators. Hence \([B_1]\) and \([B_2]\) become square matrices. Also, we select the actuators locations such that \([B_1]\) and \([B_2]\) are nonsingular. The control design can then be written as

\[
\{h_aG_1\} = -[B_1]^{-T}[C_1]\{\eta\} \quad \{h_aG_2\} = -[B_2]^{-T}[C_2]\{\xi\}
\] (44)

Note that the control design approach discussed above is but one way of designing the controller. The advantage of this approach is that it does not require a modal expansion or linearized version of the actual model. On the other hand, such an approach, while guaranteeing stability, is not globally optimal for the entire structure.

An alternate control approach, one which approaches optimality a bit more than the above approach is as follows. We linearize the mathematical model about an operating point and write the linearized model as

\[
[M]{\ddot{x}} + [K]{x} = [D_R]{F_R} + [D_E]{F_E}
\] (45)

in which \([M]\), \([D_R]\) and \([D_E]\) retain their forms earlier and

\[
[K] = \begin{bmatrix}
0 & 0 & 0 \\
0 & [K_{22}] & 0 \\
0 & 0 & [K_{33}]
\end{bmatrix}
\] (46)

We then solve the associate eigenvalue problem, which gives the eigenvalues and eigenvectors of the system. The eigenvectors can be combined into the modal matrix

\[
[U] = \begin{bmatrix}
[U_{11}] & [U_{12}] & [U_{13}] \\
[U_{21}] & [U_{22}] & [U_{23}] \\
[U_{31}] & [U_{32}] & [U_{33}]
\end{bmatrix}
\] (47)

The zero elements in the modal matrix correspond to the rigid body modes in the system. Using the transformation \(\{q(t)\} = \{q_\theta(t)\}^T \{q_x(t)\}^T = [U]\{x(t)\}\), in which \(\{q_\theta(t)\}\) denote the rigid body modal coordinates and \(\{q_x(t)\}\) are the elastic modal coordinates, the modal equations of motion become

\[
\{\ddot{q}_\theta(t)\} = [U_{11}]^T[D_{11}]{M(t)}
\] (48)

\[
\{\ddot{q}_x(t)\} + [\lambda]\{q_x(t)\} = \{Q_R(t)\} + \{Q_E(t)\}
\] (49)
in which $[\lambda]$ is a diagonal matrix containing the eigenvalues and $\{Q_R(t)\}$ and $\{Q_E(t)\}$ are the modal forces in the form

$$\{Q_R(t)\} = [U'][D_{11}]\{M(t)\} \quad \{Q_E(t)\} = [U'']\{F'(t)\}$$

(50)

in which

$$[U'] = \begin{bmatrix} [U_{12}]^T \\ [U_{13}]^T \end{bmatrix} \quad [U''] = \begin{bmatrix} [U_{22}]^T & [U_{32}]^T \\ [U_{23}]^T & [U_{33}]^T \end{bmatrix} \quad \{F'(t)\} = \begin{bmatrix} [B_1]^T\{h_aG_1\} \\ [B_2]^T\{h_aG_2\} \end{bmatrix}$$

(51)

The first term on the right of Eq. (49) describes the effect of the rigid body motion control, with the second term denoting the effects of the elastic motion control. Noting that $[U'']$ is nonsingular, we design the elements of $\{Q''(t)\}$ individually in the form

$$\{Q_E(t)\} = [C]\{\dot{q}_x(t)\}$$

(52)

in which $[C]$ is a diagonal gain matrix. The actual control input is then obtained by inverting the second of Eq. (50)

$$\{F'(t)\} = [U'']^{-1}\{Q_E(t)\} = [U'']^{-1}[C]\{\dot{q}_x(t)\}$$

(53)

and the piezo actuator inputs are obtained by inverting the last of Eq. (51). It follows that the closed loop equations for the secondary motion become

$$\{\ddot{q}_x(t)\} + [C]\{\dot{q}_x(t)\} + [\lambda]\{q_x(t)\} = \{Q_R(t)\}$$

(54)

while the equations of motion for the primary motion remain as they were in Eq. (48). Note that with this approach, the input decoupling properties between the primary and secondary motions are preserved, while there no longer is independent internal actuation among the secondary motions of the individual links. There is, however, internal actuation that involves all the coordinates associated with the secondary motion.

**Illustrative Example**

We illustrate the proposed approach by a numerical example. We consider the double link, and select each link to be of aluminum ($\rho = 2800$ kg/m$^3$, $E = 72$ GPa), length $L_1 = L_2 = 2.8$ m, with a cross section of 0.1 m by 0.01 m. For the piezoelectric actuators we use PZT with the properties of $e_{31} = 16.92$ C/m$^2$ (C: Coulombs), $h_a = 0.0001$ m, and $G_{\text{max}} = 600,000$ V/m (V: Volts). Note that 1 V = 1 kg m$^2$/C s$^2$. Hence, the maximum applicable voltage is $h_aG_{\text{max}} = 60$ V.

The control action is implemented using two torquers, one each at the pin joints, and two segmented piezoelectric actuators on each beam, of length $2L/16$ and $3L/16$ ($L = L_1, L_2$). The actuators begin at $x = 3L/16$ and $x = 9L/16$ ($x = x_1, x_2$) and they cover the entire width of the beam.
The equations of motion are too lengthy to be included here, so we proceed with discussing the control design. We consider an open loop maneuver for the primary motion and use

\[ M_1(t) = 8 \text{ Nm} \quad 0 < t < t_f/2 \]

\[ -8 \text{ Nm} \quad t_f/2 < t < t_f \]

where we select \( t_f \) as 2 seconds. For the second actuator we use the same control form with an amplitude of 4 Nm.

For the secondary motion control, we use the first control approach described in the previous section, where a decentralized approach is used in designing a separate controller for each beam. We select an ideal damping factor, \( \zeta = 0.1 \). Then, for the first link, we select the gain matrix as

\[ [C_1] = \text{diag}(-2\zeta \omega_1 - 2\zeta \omega_2 \cdots - 2\zeta \omega_n) \]

(56)

where \( \omega_s \) \((s = 1, 2, \cdots, n_1)\) are the natural frequencies of a pinned-free beam of the same dimensions as the ones under consideration, and \( n_1 = 2 \) is the number of actuators and modes included in the formulation. The idea behind this control law is to approach a uniform damping. Of course, in reality this is not possible as 1) we are dealing with a nonlinear system and 2) even if we linearize the system, the eigenvalues of the combined two link system will be different than the individual eigenvalues of each beam. Note that one can approach uniform damping by selecting the second control approach described in the previous section, but, as we will see shortly, the control approach we use gives satisfactory results. We use the same control law for the second link, with \( n_2 = 2 \).

A linearization study indicates that the natural frequencies of a pinned free beam of the same dimensions as the ones considered above are 0, 28.8 and 93.1 rad/s. The natural frequencies of the linearized model of the double link are 0, 0, 21.5, 37.5, 81.0 and 111.1 rad/s. As expected, the combined double link has lower eigenvalues than the single link.

When simulating the control action, it is possible that the control inputs generated by the control law exceed the maximum voltages that can be applied to the piezoelectric actuators. To prevent this situation, at each step of the control, the control voltages are compared to the maximum voltage and they are scaled so that all of the applied voltages are below the maximum value. The placement of the actuators and design of the control law so that the amount of scaling is reduced to a minimum is the subject of our ongoing research.

Figures 5 and 6 show the beam angles, \( \theta_1(t) \) and \( \theta_2(t) \), and the first elastic coordinate for each beam, \( \eta_1(t) \) and \( \xi_1(t) \), for the uncontrolled case. Figures 7 and 8 do the same for the controlled case. One can clearly see the coupling of these coordinates in the uncontrolled case. The proposed control approach rapidly eliminates the secondary motion. As can be seen and as expected, the secondary motion control helps stabilize the primary motion. This is because the
coupling between the primary and secondary motions transfer energy between these motions and control of the secondary motion helps stabilize the primary motion.

Note that the torque profile we use in this simulation has the goal of having zero angular motion after $t = 2$ s, so that stabilization of the primary motion is a desirable feature. In the event of continuous primary motion, energy would get transferred to the elastic motion, thus reducing the energy and momentum associated with the rigid motion. This transfer from rigid to elastic motion and subsequent loss of energy was first observed with the Explorer satellite and opened up the field of vibration control in large structures.

**Conclusions**

We have shown that, by utilizing segmented piezoelectric film actuators, it is possible to effectively control the elastic motion of flexible structures that undergo combined rigid body and elastic motion. Such control accomplishes true internal actuation. This property makes it possible to decentralize the control design, thus making it simpler to develop stable controllers. The results obtained are applicable to single body as well as multibody structures.

**References**


Figure 1: The $r$-th actuator

Figure 2: A Single Link Undergoing Large Angle Rigid Body Motion

Figure 3: A Double Link with Elasticity
Figure 4: A Spacecraft with Solar Panels

Figure 5: Angles $\theta_1(t)$ and $\theta_2(t)$ without Secondary Motion Control
Figure 6: Coordinates $\eta_1(t)$ and $\xi_1(t)$ without Secondary Motion Control

Figure 7: Angles $\theta_1(t)$ and $\theta_2(t)$ with Secondary Motion Control
Figure 8: Coordinates $\eta_1(t)$ and $\xi_1(t)$ with Secondary Motion Control